# MTH 301 Midterm Solutions

1. For groups $G$ and $H$, their direct product

$$G \times H = \{(g,h) \,|\, g \in G \text{ and } h \in H\}$$

forms a group under the operation defined by

$$(g,h)(g',h') = (gg', hh'), \text{ for all } g, g \in G \text{ and } h, h' \in H.$$

(a) Show that every line $N$ passing through the origin in $\mathbb{R}^2$ is a normal subgroup of $G = (\mathbb{R}^2, +)$.

(b) For any such line $N$ as in (a), describe $G/N$.

(c) Show that $G/N \cong \mathbb{R}$.

**Solution.** (a) As $\mathbb{R}^2$ is a direct product $\mathbb{R} \times \mathbb{R}$, it is a group under component-wise addition. Any line passing through the origin in $\mathbb{R}^2$ has an equation of the form $y = mx$, where $m$ denotes the slope of the line. As a set, such a line is given by

$$N = \{(x, mx) \,|\, x \in \mathbb{R}\}.$$

Clearly, $N \leq G$, for if $r, s \in N$, where $r = (x, mx)$ and $s = (y, my)$, then we have

$$rs^{-1} = (x, mx) + (-y, -my) = (x - y, m(x - y)) \in N.$$

To show that $N \trianglelefteq G$, observe that for any $(x', y') \in \mathbb{R}^2$ and $(x, mx) \in N$, we have

$$(x', y')(x, mx)(x', y')^{-1} = (x' + x - x', y' + mx - y') = (x, mx) \in N.$$

This implies that $gNg^{-1} \subset N$ for all $g \in G$, or in other words, $N \trianglelefteq G$.

(b) By definition, $G/N = \{g + N \,|\, g \in G\}$, so for $g = (x', y')$ and $N$ as described in (a), the coset $g + N$ is given by

$$g + N = \{(x' + x, y' + mx) \,|\, x \in \mathbb{R}\}.$$

That is, $g + N$ is the set of all points on the line $y - y' = m(x - x')$, which is parallel to $N$. In general, any line parallel to $N$ has a equation given by $(y - y_0) = m(x - x_0)$, and the set of all points on this line equals the coset $(x_0, y_0) + N$. From this, we conclude that $G/N$ is the collection of all lines parallel to $N$.

(c) Consider the map $\varphi : G \to \mathbb{R}$ defined by

$$\varphi(x, y) = mx - y, \text{ for all } (x, y) \in G.$$

Then clearly, $\varphi$ is well-defined, for if $(x, y) = (x', y')$, then $x = x'$ and $y = y'$, and so

$$mx - y = mx' - y' \implies \varphi((x, y)) = \varphi((x', y')).$$

Also, $\varphi$ is a homomorphism, as

$$\begin{aligned} \varphi((x, y) + (x'y')) &= \varphi(x + x', y + y') \\ &= m(x + x') - (y + y') \\ &= (mx - y) + (mx' - y') \\ &= \varphi((x, y)) + \varphi((x', y')). \end{aligned}$$

The surjectivity of $\varphi$ follows from the fact that for any $r \in \mathbb{R}$, we have that $\varphi((0, -r)) = r$.

From the First Isomorphism Theorem, we have

$$G/\operatorname{Ker}\varphi \cong \mathbb{R}.$$

It remains to show that $\operatorname{Ker}\varphi = N$. But this follows from the definition of $\varphi$, as

$$\begin{aligned} (x, y) \in \operatorname{Ker}\varphi &\iff \varphi((x, y)) = 0 \\ &\iff mx - y = 0 \\ &\iff y = mx \\ &\iff (x, y) \in N. \end{aligned}$$

2. Let $G$ be group and $H, K \trianglelefteq G$. Then show that

   (a) If $H \leq K$, then $(G/H)/(K/H) \cong G/K$

   (b) If $G = HK$, then $G/(H \cap K) \cong G/H \times G/K$.

   **Solution.** (a) This result is popularly known as the *Third Isomorphism Theorem*. First, note that $K/H \trianglelefteq G/H$ (which is left as an exercise). To establish the isomorphism, define a map $\psi : G/H \to G/K$ given by

$$\psi(gH) = gK, \text{ for all } gH \in G/H.$$

Then $\psi$ is well-defined for the following reason

$$\begin{aligned} g_1H = g_2H &\implies g_1g_2^{-1}H = H \\ &\implies g_1g_2^{-1}K = K \qquad \text{( since } \psi(H) = K) \\ &\implies g_1K = g_2K \\ &\implies \psi(g_1H) = \psi(g_2H). \end{aligned}$$

As

$$\begin{aligned} \psi(g_1H\, g_2H) &= \psi(g_1g_2\, H) \qquad (\text{since } H \trianglelefteq G) \\ &= g_1g_2\, K \\ &= g_1K\, g_2K \qquad (\text{since } K \trianglelefteq G) \\ &= \psi(g_1)\psi(g_2), \end{aligned}$$

$\psi$ is a homomorphism. Moreover, $\psi$ is surjective, as every coset $gK \in G/K$ is the image of the coset $gH \in G/H$ under $\psi$. Therefore, by the First Isomorphism Theorem, we have that

$$(G/H)/\operatorname{Ker}\psi \cong G/K.$$

So it remains to show that $\operatorname{Ker}\psi = K/H$, but this follows from the following argument

$$\begin{aligned} gH \in \operatorname{Ker}\psi &\iff \psi(gH) = K \\ &\iff gK = K \\ &\iff g \in K \\ &\iff gH \in K/H. \end{aligned}$$

Hence the result follows.

(b) We know from class that if $H, K \trianglelefteq G$, then $H \cap K \trianglelefteq G$, from which we can see that $G/(H \cap K)$ is a group. Put $N = H \cap K$, and define a map $\phi : G \to G/H \times G/K$ given by

$$\phi(g) = (gH, gK), \text{ for all } g \in G.$$

Note that $\phi$ is well-defined, for if $g_1 = g_2$, then

$$\begin{aligned} g_1g_2^{-1} = 1 &\implies (g_1g_2^{-1}H, g_1g_2^{-1}K) = (H, K) \qquad (\text{since } \phi(1) = (H, K)) \\ &\implies g_1g_2^{-1}H = H \text{ and } g_1g_2^{-1}K = K \\ &\implies g_1H = g_2H \text{ and } g_1K = g_2K \\ &\implies \phi(g_2) = \phi(g_2). \end{aligned}$$

Moreover, $\phi$ is a homomorphism, as

$$\begin{aligned}
\phi(g_1g_2) &= (g_1g_2H, g_1g_2K) \\
&= ((g_1H)(g_2H), (g_1K)(g_2K)) \quad \text{(since } H, K \trianglelefteq G\text{)} \\
&= (g_1H, g_1K)(g_2H, g_2K) \quad \text{(by definition of a direct product)} \\
&= \phi(g_1)\phi(g_2).
\end{aligned}$$

To show the surjectivity of $\phi$, take any $(gH, g'K) \in G/H \times G/K$. Since $G = HK$, we have that $g = hk$ and $g' = h'k'$ for some $h, h' \in H$ and $k, k' \in K$. Then then fact that $H, K \trianglelefteq G$, would imply that $gH = kH$ and $g'K = h'K$, and so $(gH, g'K) = (kH, h'K)$. Hence

$$\begin{aligned}
\phi(kh') &= (kh'H, kh'K) \\
&= (kH, h'K) \quad \text{(since } H, K \trianglelefteq G\text{)} \\
&= (gH, g'K),
\end{aligned}$$

which proves that $\phi$ is surjective.

Finally, by the First Isomorphism Theorem, we have

$$G/\operatorname{Ker}\phi \cong G/H \times G/K.$$

Furthermore,

$$\begin{aligned}
g \in \operatorname{Ker}\phi &\iff (gH, gK) = (H, K) \\
&\iff gH = H \text{ and } gK = K \\
&\iff g \in H \text{ and } g \in K \\
&\iff g \in H \cap K,
\end{aligned}$$

which establishes that $\operatorname{Ker}\phi = H \cap K$, and hence the result follows.

3. Consider the set $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ having 8 elements with an operation $\cdot$ satisfying the following relations

$$i \cdot i = j \cdot j = k \cdot k = -1$$
$$i \cdot j = k,\ j \cdot k = i,\ k \cdot i = j$$
$$(-1) \cdot (-1) = +1$$

   (a) Show that $(Q_8, \cdot)$ is a group with $+1$ as its identity element. ($Q_8$ is called the group of *quaternions*.)

(b) Is $(Q_8, \cdot)$ an abelian group? Explain why or why not.

(c) Show that every subgroup of $(Q_8, \cdot)$ is normal.

**Solution.** (a)&(b) For simplicity, let us denote +1 by 1, and for any $a, b, x \in Q_8$, we denote $x \cdot x$ by $x^2$, and $a \cdot b$ by $ab$. Since $(-1)(-1) = 1$, we have

$$o(-1) = 2 \text{ (i.e.} -1 = (-1)^{-1}) \text{ and } -1 = (1)(-1)^{-1} = (-1)^{-1}(1),$$

or in other words

$$-1 = (1)(-1) = (-1)(1). \tag{1}$$

Moreover, $i^4 = i^2 i^2 = (-1)(-1) = 1$, which implies that $o(i) \mid 4$, but since $i^2 = -1$, we infer that $o(i) = 4$. By an analogous argument we can conclude that

$$o(\pm i) = o(\pm j) = o(\pm k) = 4. \tag{2}$$

Since $jk = i$, we have $j^2 k = ji$, that is, $ji = (-1)k$, and a similar argument can be used to derive all other relations of this type, namely

$$\begin{aligned} (-1)k &= ji = k(-1) \\ (-1)i &= kj = i(-1) \\ (-1)j &= ik = j(-1) \end{aligned} \tag{3}$$

Next, we establish that $(-1)x = -x$ for all $x \in Q_8$. For the case when $x = \pm 1$, this follows from (1). So we choose $x \in Q_8 \setminus \{\pm 1\}$, say $x = i$. Suppose that $(-1)i = y.$, Then it is clear that $y \neq i$ (as this would imply that $-1 = 1$), and from (1) we see that $y \pm 1, i$, so $y \in \{-i, \pm j \pm k\}$. Suppose that $y = j$, then we have

$$\begin{aligned} & (-1)i = j \\ \implies \quad & i(-1) = j \quad \text{(from (3))} \\ \implies \quad & i(-1)j = -1 \\ \implies \quad & (-1)ij = -1 \quad \text{(from (3))} \\ \implies \quad & (-1)k = -1, \end{aligned}$$

and so $k = 1$, which gives a contradiction. Hence, $y \neq j$, and in a similar fashion we can conclude that $y \neq -j, \pm i$, and by elimination we can conclude that $(-1)i = -i$, and extending this to $j, k$ (using similar arguments), we have

$$\begin{aligned} (-1)i &= -i = k(-1) \\ (-1)j &= -j = i(-1) \\ (-1)k &= -k = k(-1) \end{aligned} \tag{4}$$

Finally, since $i^2 = j^2 = k^2 = -1$, we have the relations

$$i^{-1} = -i,\ j^{-1} = -j,\ k^{-1} = -k$$

From Equations (1) - (4), we see that $Q_8$ is closed under its operation, and every element in $Q_8$ has a unique inverse. Hence, the relations defined in the problem extend to a group operation on $Q_8$. (Note that the associativity of the operation, which has been implicitly assumed in some of the arguments above, is a cumbersome but easy exercise.) From the relations in (3) and (4), it is clear that $Q_8$ is non-abelian.

(c) Since $|Q_8| = 8$, by the Lagrange's Theorem, any proper subgroup of $Q_8$ has to be of order 2 or 4. Furthermore, any subgroup of order 4 has index 2 in $Q_8$, and hence has to be normal. So it suffices to show that every subgroup of order 2 is normal in $Q_8$.

Since any subgroup of order 2 is cyclic, it has to be generated by an element in $Q_8$ of order 2. We showed above that $-1$ is the only element of order 2. Since it generates subgroup $H = \{-1, 1\}$, $H$ is the only subgroup of order 2, so it suffices to show that $H \trianglelefteq Q_8$. For any $g \in Q_8$ and $x \in H$, we have $gxg^{-1} = 1$, if $x =$, and when $x = -1$, we have

$$g(-1)g^{-1} = (-1)gg^{-1} = -1 \in H,$$

which shows that $H \trianglelefteq G$, and the result follows.

4. We know from class that the dihedral group

$$D_8 = \langle r, s \rangle = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

is the group of symmetries of a square, which is generated by a rotation $r$ by $2\pi/4$ and a reflection $s$.

   (a) Find all subgroups of $D_8$ order 2.

(b) Show that $D_8$ has exactly three subgroups of order 4, one of which is cyclic, while the remaining two are non-cyclic. (Note that this gives an example of a non-abelian group of order 4.)

(c) Assuming that isomorphic groups possess the same subgroup structure, establish that $Q_8$ is not isomorphic to $D_8$.

**Solution.** (a) Every subgroup of $D_8$ of order 2 has to be generated by an element of order 2. The elements in $D_8$ of order 2 are the reflections $s, sr, sr^2,$ and $sr^3$, and the rotation $r^2$ by $\pi$. Hence, $D_8$ has 5 distinct subgroups of order 2, namely the subgroups generated by these elements.

(b) If a subgroup of order 4 is cyclic, then it has to be generated by an element $g \in D_8$ of order 4. Since $r$ is the only of order 4, we conclude that

$$\langle r \rangle = \{1, r, r^2, r^3\}$$

is the only cyclic subgroup of $D_8$ of order 4.

Suppose that $H$ is a non-cyclic subgroup of $D_8$ of order 4. Then by Lagrange's Theorem, every non-trivial element $g \in H$ is of order 2 or 4. If $o(g) = 4$, then $H$ is cyclic, which contradicts our assumption. Hence, every non-trivial element of $H$ is of order 2 ( $\implies r \notin H$). In other words, we have the following observation:

*Observation: $H$ has to contain 3 distinct elements of order 2.*

Before we find all such order 4 subgroups, first note that since $o(s) = (sr^k) = 2$, so we have $(sr^k)(sr^k) = 1$, which implies that

$$sr^k = r^{-k}s^{-1} = r^{n-k}s \qquad (*)$$

Using relation (*) and the observation made earlier, we can see that

$$\{1, r^2, s, sr^2\} \text{ and } \{1, r^2, sr, sr^3\}$$

are the only other order 4 subgroups.

**Exercise:** Show that both the subgroups mentioned above are isomorphic to $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. (Note that $G$ is called the *Klein-4 group*.)

(c) We proved earlier, that $Q_8$ has a unique subgroup of order 2, namely $\{-1, 1\}$. But we now know that $D_8$ has five distinct subgroups of order 2, which shows that $D_8$ cannot be isomorphic to $Q_8$.

**Exercise:** Can we conclude the same, using the structure of the order 4 subgroups?

5. **(Bonus)** Let $H = \{z \in \mathbb{C} \mid |z| = 1\}$. Then show that [10]

$$\mathbb{R}/\mathbb{Z} \cong H$$

**Solution.** First, realize that

**Exercise:** $H \leq \mathbb{C}^{\times}$.

Then we define a map $\Omega : \mathbb{R} \to H$ given by

$$\Omega = e^{i(2\pi x)}, \text{ for all } x \in \mathbb{R}.$$

Note that for any $x \in \mathbb{R}$, we can see that

$$|\Omega(x)| = |e^{i(2\pi x)}| = |\cos(2\pi x) + i\sin(2\pi x)| = 1 \implies \Omega(x) \in H.$$

We will also need to the establish the following:

**Exercise:** Show that $\Omega$ is a well-defined surjective homomorphism.

By the First Isomorphism Theorem, we have that

$$\mathbb{R}/\operatorname{Ker}\Omega \cong H. \tag{*}$$

To complete the argument, note that

$$\begin{aligned} s \in \operatorname{Ker}\Omega \quad &\iff \Omega(s) = 1 \\ &\iff e^{i(2\pi s)} = 1 \\ &\iff \cos(2\pi s) + i\sin(2\pi s) = 1 \\ &\iff \cos(2\pi s) = 1 \text{ and } \sin(2\pi s) = 0 \\ &\iff n \in \mathbb{Z}, \end{aligned}$$

which shows that $\operatorname{Ker}\Omega = \mathbb{Z}$. Hence, the result now follows from (*).